THE

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DISCUSSION ON THE REPORT ON THE TEACHING OF ARITHMETIC.

II. THE SYLLABUS IN DETAIL (p. 235).

B. COMPOUND QUANTITIES.

BY SIR G. GREENHILL.

FARMING is still an important industry in England, although neglected and depressed by comparison with Manufacture and Navigation; so it is useful still to teach the old agricultural units, mile, furlong, chain, perch, acre, rood, . . . inherited from earliest civilisation through the Roman and Greek, and based on physical unchangeable qualities, not likely to be displaced by an artificial Metric System.

Consult the lecture to the Society of Arts, December 1915, by Sir Charles M. Watson, on The Origin of the English Measures of Length, and Herodotus II 149 on Greek Measures.

But it is surprising no mention is made in the Report of the geographical units employed universally by the sailor of all countries, based on the measurement of the Earth, and the 24 hours of the day.

In these units there is never mention of the diameter or radius of the Earth, or of the military land-mile. The day being divided into 24 hours, and the equator also, longitude is reckoned in hours, each of 15 degrees or 60 minutes of time on the chronometer, and the degree is divided into 60 minutes of angle or 15 seconds of time. The length along the equator or a meridian of one minute of angle is then made the geographical sea-mile, *le mille marin*.

The Earth is then $360 \times 60 = 21600$ miles in circumference, and the quadrant from equator to pole is 5400 miles, as given in Roger Bacon's *Opus Majus*.

But in his day, and even up to the time of Newton, the distinction was not observed between this geographical mile of the sailor and the soldier's mile of 1000 paces, *millia passuum*, *mille*, reckoning with a pace of two steps, *gradus*, of 5 Roman feet, but 5·28 feet in the imperial foot as shrunk to-day.

The sailor was the cause of the confusion which led Newton astray; he had no business to use the world mile; he ought to have invented a new name, entirely different, for his geographical mile, a minute of latitude, or longitude at the equator, over the sea.

There is still the confusion in the double use of the word minute, and second, for time as well as angle.

But the landsman and engineer is introducing confusion still further, beginning to call the sea-mile a *knot*, and then using the solecism of knots an hour for speed. But this is leading us back into ancient controversy.

The sailor divides his sea-mile into 10 cable, and the cable or stadium into 100 fathom, in perfect decimal subdivision, making his fathom a trifle over 6 feet, say 6·08 feet or 6 feet 1 inch. This is so nearly 6 feet, only an inch or so over, as to mislead the school-book writer into making the fathom exactly 6 feet. But then he adds a fatal footnote—this measure, the fathom, is rarely used—oblivious of the chart and its soundings, always marked in fathom; and so destroying its independence.

The same tendency is observable in legislation; up till Edward III the foot and yard were independent measures, till an Act of Parliament made the yard exactly three feet, and destroyed the independence of one of the units.

But the rod, pole, or perch (*pertica*) of five yards and a half, cannot be cut down to fit, and must always prevail as the unit of length in ploughing, as the length of the plough and team, with 4 pole to the chain, and 40 pole to the furlong, the acre tilled by one plough being one chain by one furlong.

The fathom suits the sailor as the unit for measuring the length of a rope or sounding lead-line by the stretch of the arms, so that the fathom, *brasse*, *ὄργυια*, exists as a measure for all sailors.

Aristotle gives the quadrant of the meridian as a thousand myriad, our ten million fathom, and the quadrant is divided into ten million metres in the Metric System; so that Aristotle's estimate is nearly twice the real size of the Earth.

A Trigonometry should lead off with these geographical definitions, to add historical and physical interest to the measurements, seen on the clock or chronometer, and round the world.

G. Greenhill.

MATHEMATICAL NOTES.

488. [V. 1. a. δ.] **475.** *Math. Gazette*, July 1916, p. 296. As Professor Hill and Dr. Sheppard dispute the contention that there is no distinction in meaning between $a \times b$, $a \,.\, b$ and ab, we quote the exact words of the *Encyclopédie des Sciences Mathématiques*, I. 1. i. p. 40: "On représente un produit en écrivant d'abord le multiplicande, puis le multiplicateur, et en les séparant par un point ., ou par le signe × ou encore sans ancun signe; il est nécessaire de mettre un signe quand les deux nombres sont écrits en chiffres. Ainsi les symboles $a \,.\, b$, $a \times b$ et ab représentent chacun la somme $a+a+a \ldots +a$ de b nombres égaux à a: les symboles $23 \,.\, 12$, 23×12 représentent chacun la somme de 12 nombres égaux à 23."

We contend, in agreement with the above, that $a \times b$, $a \,.\, b$ and ab denote the same thing, viz. the product of the two factors a and b, and one uses in any given case what seems to be the most convenient of these expressions, the first being generally the best in arithmetic, and the last the best in algebra. Professor Hill apparently wishes to create a distinction, so that ab should denote a multiplication performed, while $a \times b$ is to denote a multiplication still to be performed. But surely in neither case is the multiplication performed: as long as the factors are in evidence the multiplication is merely indicated.

We are, however, not concerned with new interpretations. The whole question before us is whether $a + b \times c$ is or is not ambiguous under present conventions, though perhaps we ought also to consider the contention of Dr. Sheppard that, if under present conventions there is no ambiguity, it is desirable that the convention should be dropped so as to create an ambiguity.

We showed in our first memorandum that there is a well-established rule, given prominently in text-books, stating that in such cases multiplications (and divisions, if any) must be performed first, before it is possible to add the terms together, and we endeavoured to show that the object (or, at any rate, the effect) of this rule was to bring the conventions of arithmetic into accord with those of algebra.

With regard to Dr. Sheppard's communication, he admits most emphatically the existence of the rule, and pleads for its abrogation, so that in future such expressions shall be held to be ambiguous, although up to the present time they are free from ambiguity. His endeavour is to persuade the whole mathematical world to commit the folly, not to say the iniquity, of *creating* an ambiguity where there was none, and chaos where previously was well-established order. It is so monstrous that it seems needless to say any more about it: *mole ruit suâ* !

Professor Hill drags a red herring over the trail by discussing the evaluation of a *term* in which division as well as multiplication occurs. This is an important matter, but quite sequent to that of the existence of terms at all: our first memorandum showed clearly how we should deal with it. The first thing is to admit the existence of terms, viz. quantities which are separated from each other by + or − signs, and it is this existence which Professor Hill doubted, and which Dr. Sheppard admitted but desired to destroy.

Both Professor Hill and Dr. Sheppard seem to discuss the matter as if the question of founding a new convention were under discussion, whereas in every civilised nation in the world, and perhaps in Germany too, the convention that $a + b \times c$ means a added to the product of b and c has been established firmly for more than a century.

May it not be possible that the widespread ignorance of the principle of dimensions, which is the deepest and commonest defect in English mathematical education, is connected with insufficient preliminary discussion as to what is meant by a term in an expression? A. Lodge.

C. S. Jackson.

My friend Mr. F. J. W. Whipple makes what appears to me to be an

important suggestion, that the sign × of multiplication is usually made much too large by printers. C. S. J.

489. [V. 2.] *On a Review.*

To my review of Messrs. Richardson and Landis's *Fundamental Conceptions of Modern Mathematics* in the current number of the *Gazette*, I should like to add two notes, to which Mr. Richardson has called my attention.

First, when I said: "I heartily agree with the authors in what they say of De Morgan on pp. 120-121," I might well have quoted the authors' words: "It is a crying shame that the University of Cambridge . . . has not yet set her press to the work of issuing an edition of the collected works of Augustus De Morgan, one of the greatest of her sons." As the review stands, no one on reading it can tell whether that in which I agree with the authors is in praise or in blame of De Morgan.

Second, when I said: "In the closely allied notions that numbers are signs, that the principle of permanence is sufficient to extend our idea of number, and that a 'variable' means what it is usually said to mean, the authors seem to be correct," I (unintentionally of course) used language which is quite certain to give cursory readers the impression that the authors of *Fundamental Conceptions of Modern Mathematics* affirm numbers to be signs, the principle of permanence to be sufficient, and "variable" to mean what it is usually said to mean. They, in fact, deny each of these assertions.

PHILIP E. B. JOURDAIN.

490. [V. 1. a. δ.] **474.** *Math. Gazette,* July 1916, p. 296. "Pseudo-accuracy" is even wide of the mark in saying that the third significant figure in $(23{\cdot}7\times 0{\cdot}315)^2$, on the assumption that the given numbers are correct to three significant figures, cannot be relied upon; for all that we are then told is that the number lies between 56·14 and 55·33; thus it cannot be relied upon for even the second figure!

Apropos, what would be the examiner's answer to this little question?—"State the value of x, giving as many significant figures *as you can rely upon*, when

$$224{\cdot}4<x<225{\cdot}1."$$

J. M. CHILD.

491. [V. 10.] *What every Mathematician needs.*

It is of great importance to British mathematicians at the present time that an account of all recently published mathematical literature should be published outside Germany. The *Revue semestrielle des publications mathématiques* is published by the Mathematical Society of Amsterdam, and appears twice a year within a few months of the articles it reviews. A short account of each article is given, and no attempt is made at criticism. The list of periodicals which are regularly abstracted seems to be quite complete. References are given to all the reviews of mathematical books which appear in current literature, and this occasionally is an advantage. Both in this reference to all reviews and in the prompt appearance of accounts of articles, the *Revue* has a great advantage over the *Jahrbuch über die Fortschritte der Mathematik*, which is published in Berlin two or three years after the appearance of the articles of which it gives an account. The *Jahrbuch* is expensive, and at present is not being published. The subscription to the *Revue* is only 7s. per year, post free, and it is absolutely indispensable to all mathematicians who wish to form an idea of what work is being done at the present time. It may now be obtained in England from Mr. D. J. Bryce, 149 Strand, London, W.C.

P. E. B. JOURDAIN.

492. [V.] *"Thanking you in anticipation . . ."*

From *School and Society* we extract the following letters, which may be interesting to those who collect psychological material of the kind, or to

young aspirants, not satisfied with the state of life into which it has pleased the powers that be to call them, and endeavouring to express in language of appropriate moderation their claims to a wider recognition of natural gifts as yet unsuspected by the community in general.

The *Yale Alumni Weekly* prints the following application for a professorship at the university:

Yale Colledge—I have a natural gift of natural science. I have had no training as a professor but with the Athorities of the Colledge I could applay a great value. I would like a good paying position like that. To test me ask me any question wich is supposed to have no answer try me by mail well enough or through the Warriors Club. found in old directories so that my trip will be quiet certain ask me anything in regards to the relations of one thing to another also of accidents or the caracter of the life of anything from the winds to the human being or the spirituel to the matearial of all supersitions of any part of the body as thought builds it I will give them to you with prove. I can regulate inclinations by diet cure decease.—Sincerely.

The following is from Georgia, and is a worthy pendent to the above:

Dear Sir as I understand the matter, that you will furnish aid or, will help a worthy project your Hon, Sir, I am going to make known to you a secreate that I have not made known to others or, to non other than my own people, or famly. The Secreat is simply this. I am a geanis, and I have underway 3 great projects, one the greatis that has ever leaped from the Brain of man, this last one menshion has many diferance action or feuturs which is difficult for me to carry out on account of not having surficient tools nither have I time to work at the project because of my financil strain, thanking you very much for your favor to reply, in favor to help me in this matter or to reject.—Yours truly.

REVIEWS.

Report on Gyroscopic Theory to the Aeronautical Committee, 1914. By Sir George Greenhill. Printed by the Stationery Office, and sold by Wyman & Sons, Breams Buildings, Fetter Lane, E.C. Price ten shillings.

A collection is made in this report of the scattered theory and formulas of the motion of a spinning body, such as the top, and extended to other familiar examples of the gyroscope, great and small, and introductory to the practical applications arising to-day in the steering and stability of the flying machine and airship, the gyro-compass, the single track gyroscopic car, and in the whirling effect of a revolving shaft carrying a flywheel, as in the Laval turbine. On the largest scale the theory is investigated of the Precession of the Equinox, revealed to us by astronomical observation, and illustrated too by the motion of the Moon's Node.

The mathematical treatment is carried out with analytical completeness, as far as possible in the mathematical development of to-day, to make the report serve for a collection of reference when the theoretical investigation is required without delay when a practical problem is proposed, requiring a solution within the limits of practical requirement.

The distinction arises at the outset between dynamical and statical equilibrium and stability.

In statical equilibrium the centre of gravity of a system seeks out the lowest position it can assume. But in the dynamical equilibrium, for instance of the sleeping top, the centre of gravity rises as high as it is able; and when the top is steady upright, it is said to sleep.

A man or animal sleeps lying down in statical equilibrium, with the centre of gravity at rest in the lowest position.

But in movement a man assumes the noble upright attitude of the biped, or rides upright on the back of a horse, a camel, or a bicycle, in the position of statical instability.

This is for ease of progression, and any burden, rifle or knapsack, he prefers to carry as high up as possible. Mounted still higher on stilts, or as Blondin on a tight rope, his progress is not more difficult, with the confidence of experience.

A confusion of statical and dynamical stability has led to serious mistakes and misapplication of theory. Such as ballasting a ship too low, making it bottom-heavy and uneasy among waves, according to the recommendation of Bernoulli the Swiss admiral, very good on a lake, but uninhabitable at sea; also in the medical doctor's design of a soldier's knapsack low down on the back, suitable only for a halt, on the march fatiguing and uncomfortable.

Then there was the craze for spreading the railway gauge so as to lower carriage body and boiler between the wheels.

Such theory has found itself in conflict with practical experience, realised we see to-day in the modern tramcar and motibus, as well as in the old prints (there is a good illustration in Verdant Green) of the stage-coach loaded with passengers and luggage high up on the roof, and running freer in consequence with a slower roll on the springs.

And to-day the modern locomotive has the boiler pitched as high as possible to pass under a bridge.

It is the limitation of the loading gauge of bridge and tunnel of the timid early railway engineer, brought up in the traditions of canal work, which has hampered railway development, and not so much the restriction of the gauge of the rails, the original Roman gauge of 5 feet outside, adopted all over the world; and 7-foot gauge is all pulled up; while the megalomania of the Irish gauge, 5 feet 6 inches, has not spread out of that island; any tendency discernible so far is towards a reduction of gauge, especially in the delightful toy railways, to open out picturesque scenery.

Practical experience has not borne out the early theory of Brunel and Crampton, statical and misapplied, that a low centre of gravity is conducive to steady easy running of a locomotive or railway carriage on springs; and their high single driving wheels, symptom of the small boiler scant of breath, have disappeared to-day with the increase in boiler power to cope with the heavier trains.

A lecturer on gyroscopic theory with experiments will do well to avoid the use of string to spin a top. A glance of the audience at the length of his string will show if the lecturer is going to foozle his spin.

The length should not exceed half the stretch of the arms, as this is well known to the cheapjack as the place where the relative velocity of the hands is a maximum, a fundamental principle too in boxing.

Better use some apparatus more visible to the audience than a top; and a bicycle wheel, ready made to hand, is excellent for the purpose, with the axle prolonged into a stalk; this can be spun by hand in a smooth cup on the table; and it can be taken up and brandished to feel the muscular sense of the gyroscopic reaction, and so imitate the feelings of the flying machine, when making a turn or circle, due to the revolving screw.

Hold the stalk in a fixed position, with the axle horizontal or inclined, and the wheel makes an excellent pendulum if put out of balance by a bar between the spokes; the angle of oscillation may be made as large as desired, or the wheel may make complete revolutions to illustrate pendulum theory. Or else the bicycle complete may be stood on its back on the table, and the front wheel used as a pendulum put

out of balance; while the back wheel gives the effect of rotation on an axle not a principal axis.

To show off complete gyroscopic motion the bicycle wheel must be suspended to hang freely by the stalk in a vertical position. Alt-azimuth suspension can be given from a bicycle hub on an iron bracket bolted to the under side of a beam or sleeper; or a mere swivel hook will serve with the stalk hooked on to it to swing and revolve freely in altitude and azimuth. Spin enough can then be given to the wheel by hand.

As the motion is in three dimensions some such apparatus is essential to visualise it, and preserve the distinction of right and left, up and down, with the clock and against it, deasil and widdershins, distinctions impossible to keep clear in the Flatland of a blackboard drawing, or on a sheet of paper.

The first chapter of the report is restricted to steady gyroscopic motion, and an exact geometrical treatment is given with as few symbols as possible; and the same geometry can be extended to a number of allied problems, as of the steady motion of plate, dish, egg, or wine-glass on the table, or of a stud or coin rolling on the floor; also of a wheel, bicycle, or motor-car on a road or banked track.

Kelvin's rule—Hurry the precession and the top rises—can then be extended to the bicycle to explain why it must be steered into a smaller circle to remove the upright position, and how the stability is maintained by incessant attention to over-correction.

This over-correction is made automatically in the top or gyroscope, so that a slight tremor of nutation is always present, even if invisible in the motion apparently steady. The slight nutation is evidence of the stability of the motion, and is capable of elementary treatment. But when the nutation becomes visible and considerable, the treatment requires the Elliptic Function in all its complexity, and Chapter III gives it without passing over any difficulty. The history of the general unsteady motion of the axis of the top makes an interesting chapter in showing the influence of dynamical requirements on analytical development in creating the theory of the Elliptic Function.

This function had not been invented 150 years ago when Euler first undertook the problem of gyroscopic motion. Euler identified the motion of the axis in altitude or nutation with the beat of a pendulum swinging through a large angle, like the bicycle wheel. This pendulum beats the Elliptic Functions of the first kind, and Euler's idea is still useful for giving a concrete illustration of these functions. But the motion in azimuth brings in the Third Elliptic Integral, and theory had to wait another sixty years for the treatment by Legendre, and then only of the complete integral required for the apsidal angle.

Some years later the Theta Function was invented by Jacobi for the expression of the Third Elliptic Integral, incomplete or complete. But the mixture of real and imaginary argument in the application to the top renders the expression useless by the Theta Function, although analytically complete.

As the Gyroscopic Report is intended to serve for reference when a practical problem arises, no details have been passed over in this Chapter III required in a complete treatment.

The dynamical constants are three in number in a state of top motion, making the choice of an illustrative numerical example an embarrassing one in its variety.

But some representative diagrams must be selected carefully to illustrate an actual case of motion; here Abel's theory of the pseudo-elliptic integral comes in useful, in which the theta function quotient is made algebraical by the selection of the elliptic parameter a simple

aliquot part of a period, and so makes an oasis in the desert of the general theory.

By the choice of another constant the motion may be made to close in on itself and be expressible algebraically. But in an experiment this state must not be expected to persist, as a very slight amount of friction is enough to mask the features in changing the apsidal angle, as in the double pendulum tracings.

The simplest cases of such algebraical motion are discussed in Chapters V and VI, and illustrated by the appropriate diagram.

Otherwise Darboux's theory in Chapter IV of the associated deformable hyperboloid can be used to replace the analytical constants by geometrical lengths; calculation is thus replaced by measurement on a drawing.

Chapter VII discusses the Spherical Pendulum motion, realised when the gyroscopic apparatus is projected without any rotation of the bicycle wheel.

Chapter VIII shows the application of moving axes to the motion of a rolling sphere, and of a body alive inside with flywheels, or vessels of liquid, like an airship or flying machine.

The report concludes in Chapter IX with problems of rolling bodies, intractable at present in the general state of unsteady motion; but susceptible of treatment when steady, with a slight tremor superposed of nutation.

In this way the stability may be discussed of a hoop rolling along upright in a straight line, as well as of the more general classical problems, due chiefly to Puiseux. The treatment finds a place here of the gyro-compass, and of the rotating shaft carrying a flywheel, in its stability against whip, as for instance with a screw propeller pitching out of water, and flying round unsupported at the end of the shaft.

Hitherto the last word on gyroscopic theory has been found in Routh's *Advanced Rigid Dynamics* and the *Kreisel-Theorie* of Klein-Sommerfeld-Noether; and this report will serve to coordinate and extend the treatment and applications. G.

Theory and Applications of Finite Groups. By G. A. Miller, H. F. Blickfeldt, and L. E. Dickson. Pp. xvii + 390. 17s. net. 1916. (New York, John Wiley & Co.)

It is rare to find a treatise introductory to a branch of higher mathematics which is divided into parts, each written by an expert who has interested himself in the special subject of which he writes. This is the idea of the present work. The advantages are obvious enough. We might have anticipated corresponding drawbacks; some lack of unity perhaps, or a tendency on the part of each writer to revel in a mass of detail taken from his own researches, which is doubtless interesting enough to the author, but hardly suited to the needs of the student. However, the temptations seem in the present work to have been resisted. The close contact between the authors and their mutual criticism of each other's work has secured a reasonable balance and connexion between the parts; and the emphasis laid on each author's own work is only sufficient to give a personal interest, and has not involved sacrifice of important matter.

The question of nomenclature is always a difficult one in group-theory. The authors take "substitution" as meaning "permutation," and "transformation" as meaning "linear substitution"; which seems satisfactory enough. We may also note "co-set" as equivalent to "Nebengruppe" or "partition," "cross-cut" for "greatest common subgroup" or "Durchschnitt," and so on. The definition of "product" of two transformations is not that usually adopted in this country: the point is discussed on p. 197. The same applies to the "canonical form" of a linear transformation, though the two definitions agree for transformations of finite order, which are alone discussed in the present work.

Part I., by Prof. Miller, deals with permutation-groups and abstract group-theory. Concrete examples of groups, especially permutation-groups, are given, and gradually prepare the student for the abstract ideas of group, isomorphism, quotient-group, etc. This method has great advantages; but, though it is doubtless advisable that a student should first meet a new idea in a practical application, formal definitions should be given and emphasised sooner or later. I cannot find a rigorous definition of such terms as "power" (contrast p. 196; but not in index), "quotient-group," etc., which would quite suffice for a novice. The style in places might be made less obscure by more plentiful use of commas, or more detailed explanations. But the subject of groups is not an easy one, so we must not be too critical. A little assistance from the teacher should supply all deficiencies. The abundant references and historical notes form a pleasing feature of Prof. Miller's work. Some examples to be worked by the student are given here (and in the other two parts), which are quite suitable for their purpose, with suggestions for the solution in the harder cases.

Part II., by Prof. Blickfeldt, deals with linear transformation-groups and their invariants, the groups of the regular polyhedra, primitive groups, and group-characteristics. The work is well done. A combination of this part with the corresponding chapters of Burnside's *Theory of Groups* will suffice to put the reader in contact with all that is really essential in this subject.

Part III., by Prof. Dickson, does not deal with the work which the author has made especially his own, and which will be found in Dickson's *Linear Groups*. It contains instead the application of group-theory to the solution of algebraic equations. The author has dealt with the matter also in his *Theory of Algebraic Equations*, but it is quite well worth having his most recent ideas on the subject. Chapters are added on ruler-and-compass constructions, inflexions of the plane cubic curve, Geiser's theory of the connexion between the cubic surface and the plane quartic, etc. The reader may supplement this by Part II. of Weber's *Algebra*.

The authors may be congratulated on an enterprising venture not unworthy of the eminent mathematician to whom the book is dedicated.

HAROLD HILTON.

Fermat's Last Theorem. Rigid proof by elementary algebra, also dissertation on test for primes, and recurring decimals. By M. CASHMORE. Pp. 63. 2s. net. 1916. (Bell & Sons.)

This book opens with some "preliminary considerations," from which it appears that the author thinks that $x=\frac{1}{2}$ is a solution of $\frac{(5-3x)(1-2x)}{1-2x}=1$.

It is, of course, not surprising that he has failed to prove Fermat's Last Theorem.

The only "test for primes" that pretends to be original is stated inaccurately. The meaning is, however, apparent from the sequel, and the result is perfectly obvious. The remarks on recurring decimals might be readable if they were re-written in about a tenth of the space.

The style throughout leaves much to be desired. A. R.

An Introduction to the Use of Generalized Coordinates in Mechanics and Physics. By W. E. BYERLY. Pp. vii + 118. 5s. 6d. 1916. (Ginn & Co.)

The author first establishes the Lagrangian equations of motion for a single particle, and then extends them to the general case. In the second chapter he deals with the Hamiltonian equations and with Routh's modified Lagrangian expression for the kinetic energy of a system, and with the method of ignoration of coordinates. The third chapter deals with impulsive forces, and the fourth with conservative forces and the principles of least action.

The arguments and results are put very clearly, the latter are illustrated throughout by an excellent set of worked examples of varying degrees of difficulty, and there are also sets of examples for practice. It is the simplest and most satisfying exposition of the subject that we have seen.

We have noticed no serious misprints: p^2 for p_2 on p. 39, l. 2, and a curious printer's error in the heading of pp. 77, 79, which should be "Problem in Fluid Motion." On p. 39 a reference to p. 35 for the definition of p_k would be of assistance, as without it the introduction of p_1, p_2 . . . looks a little abrupt. On p. 62, l. 6 from the foot, we are sorry to see the term "impulsive force" used instead of "impulse," and in the sequel, "forces" are mentioned where obviously "impulses" are meant. Also, p. 63, art. 28, "an impulsive force of magnitude P" would be better stated as "a sudden impulse of magnitude P," or "an impulsive force whose impulse is of magnitude P." The magnitude of the force itself is, of course, infinite, and is moreover of different dimensions from its impulse. The term "blow," as equivalent to "sudden impulse," is very happily used in (*d*), p. 66. A. LODGE.

Historical Introduction to Mathematical Literature. By G. A. MILLER. Pp. xiv + 302. 7s. net. 1916. (New York: The Macmillan Company; London: Macmillan and Co., Ltd.)

It is not uncommon in America to use such phrases as "*the* mathematical literature" (p. v and elsewhere) and "*the* analytic geometry" (p. 185), and presumably a patriotic Briton would not object to this, for the custom is French as well as German. To one who knows these languages, American does not present great difficulties, and this is as well for mathematicians, as so much mathematical work is done in the States nowadays. The use of the word "overlook" in the sense of taking a general survey is very German—and not always inappropriate. The main object of the present work is (p. v) to lead the mathematical student "to points from which he can overlook domains of considerable extent in order that he may be able to form a somewhat independent judgment as regards the regions which he might like to examine more closely." Thus, the seven chapters and the Appendix of this book deal with such subjects as the changes in point of view during the nineteenth century with regard to the place of mathematics and its history in education, the history of mathematics in America since about 1870, mathematical societies, congresses, journals, encyclopaedias, and other works of reference, definitions and dominant concepts of mathematics, errors in mathematical literature, mathematics as an educational subject, fundamental developments in arithmetic, geometry, and algebra, and short sketches of the lives and work of various prominent deceased mathematicians.

The book will be useful to those who teach: learners require something more than personal touches and corrections of errors in other books—chiefly on the theory of groups, and a little "selected" information on the history of mathematical ideas. What a student needs is a detailed and interesting description of the origin and growth of ideas; such anachronisms as a mention of the integral calculus when the work of Archimedes is spoken of must, as Prof. Miller points out on p. 14, be avoided; and it must constantly be remembered that "a great variety of motives—religious, economic, aesthetic, and philosophic—have contributed to the development of mathematics" (p. 22).

Coming to details, the statement made by Descartes in 1640 and quoted on pp. 8 and 59, is, as stated on the latter page, from the *Philosophical Collections*. It is rather surprising that the only reference given to it on the page first mentioned should be to presumably a quotation of it in the *Philosophical Transactions* (*Abridged*) of 1809. On p. 289, De Morgan's name is spelt with a small "d": De Morgan always objected to this. There is an unfortunate wording that the Greeks proved that "the square root of 2 cannot be a rational number": the truth is that, for valid logical reasons, they did not admit that "irrational quantities" could be put in the same class as "numbers" (cf. pp. 134-136). I do not think the author's view is correct that "this ancient Greek view impeded mathematical progress during two thousand years until Descartes exhibited the deep unity of these two fields" (p. 136; cf. p. 140). The Greek view happens to be logically correct, and Descartes' advance was owing to the fact that he overlooked fine distinctions for the sake of an analogy. Certainly a learner must get more into sympathy with Pytha-

goras and Descartes than with Plato and Euclid, but a mathematician must appreciate logical difference as well as analogy.

It is only needful to mention that there are very many useful features in this book (for example, the references on pp. 7-8), and that the errors do not greatly interfere with the usefulness.

A Course in Mathematical Analysis: Functions of a Complex Variable: being Part I. of Volume II. By ÉDOUARD GOURSAT. Translated by E. R. HEDRICK and O. DUNKEL. Pp. x+260. 11s. 6d. net. 1916. (Boston, New York, Chicago, and London: Ginn and Co.)

The same translators have already published a translation of the first volume of the first edition of Goursat's *Cours d'Analyse*. In the second French edition the first volume was not altered so radically as the second one; and hence the translation of the first part of the second volume, which has been made from the second edition, may be used in conjunction with that of the first volume. However, references are given in the book under review to both editions of the first volume.

This Part is occupied with the theory of analytic functions, and a second Part will deal with differential equations. In the Part under review, we have a treatment of the general principles of the theory of analytic functions, power-series, conformal representation, and Cauchy's theory of complex integration and its applications to the theory of power-series, residues, periods of definite integrals, and so on. The third chapter deals with one-valued analytic functions according to the methods of Weierstrass: the theorems of Weierstrass and Mittag-Leffler, Weierstrass's theory of elliptic functions, and inverse functions. The remaining chapters deal with analytic "extension" (for which the word "continuation" is more usual), natural boundaries, and analytic functions of several variables.

The book does not begin in a very enlightening manner: a complex number "is any expression of the form $a+bi$ where ... i is a special symbol which has been introduced in order to generalise algebra" (p. 3). But this is only a specimen of the usual fashion in which text-books hurry over the fundamental parts of a subject, and, I suppose, presuppose that "good grounding" which hitherto I have looked for vainly in our education. On pp. 7-9, the term "analytic function" is defined in the correct fashion, but no indication is given as to the essential part played by proofs of Cauchy's theorem in deciding whether this definition should include continuity of the derivative or not. Such a discussion would be of immense value for the student, and it is well known that the proof of Cauchy's theorem without such a supposition is one of the most striking parts of the work of Goursat himself. Definitions in a text-book are simply signs that the conceptions defined have been or are important in the development of mathematics, and it is only by studying the history of the development that we can form any idea of what this importance is. Goursat's proof of Cauchy's theorem is given on pp. 66-69, but it seems that the "nerve" of the proof is rather hidden from a student: it would have been worth while to develop the subtleties of this proof singly and at far greater length. It is pleasant to see, on p. 78, the proof of the converse of Cauchy's theorem due to Morera, which may be taken to be, from another point of view, as fundamental in the whole theory as Cauchy's own theorem is from the usual point of view. There seems to be, on p. 139, an implication that Cauchy had anticipated Weierstrass's product-theorem: "Long before Weierstrass's work, Cauchy had deduced from the theory of residues a method by which a function analytic except for poles may ... be decomposed into a sum of an infinite number of rational terms." It is very often not realised that the theorems of Weierstrass and Mittag-Leffler prove the *existence* of an analytic function with certain assigned zeros or poles; but that Cauchy's method has the quite different object of development of a *given* function in a particular form. Thus, it is not an application of Weierstrass's theorem to develop $\sin x$ in a product, though it is an application of Cauchy's method.

The book is very well translated and printed, and will be found very valuable for a foundation of courses of lectures. PHILIP E. B. JOURDAIN.

Arithmetic for Engineers. By C. B. Clapham. Pp. 435. 5s. 6d. net. 1916. (Chapman & Hall.)

This volume is one of the D.U. ("directly useful") series issued under the editorship of W. J. Lineham. In addition to the subject-matter of the title, it contains simple algebra, mensuration, graphs, logarithms, etc., such as is suitable for junior students. The author makes the just complaint that there is not a text-book that is really suitable for the needs of the technical student; they are either far too theoretical, or far too so-called practical, *i.e.* scamped. He therefore proceeds to write a text-book that suits his own ideas, and in doing so has managed to write a text that will be found to meet the ideas of a great many teachers who have felt the lack of a suitable introduction to technical mathematics. The following minor points, however, seem to me to mar an otherwise excellent work.

The volume opens with vulgar fractions, treated from the material standpoint; and in consequence the proof of the rules for multiplication and division of fractions is laboured and, to me, unconvincing. The next chapter on decimals is done thoroughly well, although there are several controversial points that occur; for instance, the direction to insert the decimal point in a quotient as the corresponding figure in the dividend is "brought down," and the introduction *afterwards* of the use of rough checks; the method of which, by the way, the author believes is not generally known; here it would have been better to have opened the chapter with the *well-known* notation of "Standard Form," although, a point unnoted in this book, this idea is unsuitable for making rough checks in expressions in which square and other roots occur; further rough checks need only be rough, thus, a check of 40 is quite enough to separate 63·7, say, from either 6·37 or 637, and the idea of getting an accurate rough check is illogical and waste of time. It is good to see, in this and the following chapters, that the author adheres to "significant figures" instead of the absurd "so many decimal places"; and further, that he does not give such ridiculous results as a five-figure quotient as the result of a division of a four-figure number by a three-figure number. It is time some one gave us four-figure logarithms instead of four-place logarithms, such as the tables at the end of the book, which seem to be those that are issued by the Board of Education.

The thermometer is used to introduce negative numbers, but is made far too much of; if the author, having first introduced the idea in this way, had proceeded to use two scales, with zero at the middle of each, see what an opportunity he would have had of giving the theory of the slide rule. In multiplication by a negative, it must be carefully observed that the author, in defining $a \times b$ as b multiplied by a, or a times b, is not following the usual convention; even then he has no logical reason for his statement that "$(-3)\times(-7)$ must mean -7 *subtracted* three times from 0, since the sign of $+$ and $-$ have directly opposite meanings."

In the chapter on Simple Equations, there occurs the usual fallacy in 'practical' text-books, such as

$$\text{“ } 3w = 168, \text{ therefore } w = 56 \text{ lbs.”}$$

With this kind of thing excepted, this chapter is developed in a manner that cannot but prove exceedingly valuable to students using the book—more especially as it is immediately followed by that which is usually (most unwisely) neglected, namely, a discussion of Transposition of Formulae. Like all the other chapters, a great feature of the book, this is profusely exampled with questions taken from practical every-day experience in the laboratory and the shop, there being no faked riddles to solve.

For 'practical' juniors, it is far preferable to neglect entirely the connection between logarithms and indices; it is well-nigh impossible for such a student, even with the best of teaching and the proof that 2^{10000} lies between 10^{3010} and 10^{3011}, or the equivalents, as given in Prof. Bowley's book, to appreciate the meaning of an irrational index, or even of the decimal approximation thereto.

The book closes with a large section on mensuration and an introduction to graphs. The latter is diametrically opposite to my views and also to my experience. The first, and the only thing that a 'first-year' student should

learn, is the straight line graph and *its use as a measure of rate*, thus introducing the idea of the calculus unconsciously.

The slide rule is well explained, but I feel that this instrument should always be the subject of personal tuition, when not more than twenty minutes is necessary at the outside to explain the main principle, after which the rest should be found out by the student himself.

As I said above, these are all minor points; the book is a praiseworthy attempt to give the requisite amount of theoretical training in a practical manner, or rather to teach practical mathematics theoretically; nothing else matters. Every teacher should accept this critique of mine with such reserve that, if he is dissatisfied with the texts he has so far seen, he will feel compelled to criticise the book for himself at first hand; that is the best advice I can give him. J. M. CHILD.

The Mathematical Theory of Probabilities, and its Application to Frequency Curves and Statistical Methods. By A. FISHER. Pp. xx + 171. 8s. 6d. net. 1916. (Macmillan.)

This is a translation from the Danish of the first part of Mr. Arne Fisher's volume, and covers the sections on Mathematical Probabilities and Homograde Statistics. The close connection between economics and the science and art of statistics has given of recent years a considerable stimulus to research in the theory of probabilities. Official bodies in this country have begun to recognise, perhaps even yet but dimly, that their work is not completed when the public is confronted with volumes of blue-books filled with pages of tables. To accumulate and to tabulate the data makes, comparatively speaking, no excessive demand upon the intelligence of the stereotyped class of official, the admired of Commissioners, drawn in the main from those to whom the "damned dots" are *anathema maranatha*. Their previous training in classics and philosophy—beyond all else, most of them have an "equal distribution of ignorance"—has long been supposed to be adequate for such a simple purpose as to enable them to give off-hand and without a blush a definition of, for instance, "equally likely," that will at any rate satisfy—themselves. But those days are rapidly passing. We have already seen in the pages of the *Gazette* an instance of the "intrusion" of the Integral Calculus into the mysteries of the Income Tax, and now the highest resources of the mathematician may be called upon for decisions connected with such subjects as the expenditure of the various classes of the community, counting the tail fin rays of flounders, or the conservation of the supplies within the reach of a nation. We are, however, a long way off from having the central thinking department in statistics demanded by such authorities as Prof. Bowley. The reason is partly due to the inertia inherent in the situation, indicated above, and until recently the official who gloried in his generally unintelligible tables, might not unfairly scoff at the mathematician and tell him, as the monarch told the herald, that he did not even know his own silly business. For it is a curious fact that although the names of De Moivre, Stirling, Bayes, De Morgan, Crofton, and Whitworth on the one side, and those of Mill, Venn, Keynes, etc., on the other, remind us that this country played its part in the mathematical and philosophical development of the subject, yet no work in English is to be found which embodies the results of *modern* research. For that we must go to Scandinavia, Denmark, Russia, Italy, or Germany, although the names of Pearson, Edgworth, and Yule stand like milestones along the path of progress. Such an effort as that of Mr. Fisher, to "treat all the modern researches from a common point of view based upon the mathematical principles contained in the immortal work of the great Laplace" will be heartily welcomed. In his preface, he states that it has been his aim to "present a theory of probabilities . . . which would prove of value to the practical statistician, the actuary, the biologist, the engineer, and the medical man, as well as to the student who studies mathematics for the sake of mathematics alone." The treatment throughout is extremely clear. A signal instance is the chapter on "Probability à Posteriori," in which the source of many errors is traced to the unhappy term of De Morgan, inverse probabilities, in which Bayes receives his full due, and the limitations of his famous rule,

still considered "theoretically true," and capable of giving correct results "if we are able to enumerate and determine the probabilities of existence of the complexes of origin," are closely scanned. We look forward with pleasant anticipations to the rest of Mr. Fisher's work. But let him not forget an Index, and let him see to it that the unpleasant *D'Alambert*, repeated again and again in his pages, disappears in future references to "one of the ablest thinkers of his time."

Euclid's Book on Divisions of Figures, with a Restoration based on Woepcke's Text, and on the Practica Geometriae of Leonardo Pisano. By R. C. Archibald. Pp. viii + 88. 6s. net. 1916. (Cambridge University Press.)

But for the short section 3, pp. 8-10, in Heath's great edition of Euclid, it is more than probable that a considerable number of our readers would be unaware that Euclid had written a work, *περὶ διαιρέσεων*. Proclus (*v.* chap. v. of Taylor's *Commentaries*) tells us that "there are many other mathematical volumes of this man, full of admirable diligence and skilful consideration: for such are his Optics and Catoptrics: and such, also, are his elementary institutions, which conduce to the attainment of music; and his book concerning divisions." To which Taylor adds in a note, "This work is most probably lost. See Dr. Gregory's Euclid." And later, writing on Euclid's definition of a figure, we find: "A circle and every right-lined figure may be divided by reason or proportion into dissimilar figures; which is the business of Euclid in his book of divisions, where he divides one figure into figures similar to such as are given; but another into such as are dissimilar." For eighteen centuries no more than this seems to have been known to the western world of the *Book on Divisions*. But 1570 turned out to be a notable year in the history of the works of Euclid. It saw *The Elements of Geometrie*, ascribed to "the most auncient Philosopher Euclide of Megara," translated "into the Englishe toung" by no less a dignitary than a Citizen and Lord Mayor of London, and adorned with a "very fruitfull Praeface" by Dr. John Dee. At this stage in Dee's remarkable career his mathematical achievements were much to the fore. Twenty years before, when he was at the mature age of twenty-three, crowds of eager students clustered like bees around the outside of the room at the College of Reims, to hear the English visitor to their renowned Paris discourse as had never been done in any University in Christendom, "Mathematicé, Physicé, et Pythagoricé." In 1563, when on a visit to Commandinus at Urbino, Dee presented his distinguished host with a copy of a treatise, a *Liber Divisionum* by Machumetius Bagdedinus. There is unfortunately no mention in Dee's Diary of the catalogue he made in 1583 of his precious MSS., or we should have no doubt many remarks couched in the vein of delightful reminiscence, or in that of quaint reflection which distinguished the "fruitful Praeface" above alluded to. Be that as it may, Commandinus, "at the request of J. Dee of London" (say John Leeke and Geo. Serle, students in the Math., in their edition of Euclid of 1661) published this Latin translation in 1570. Dee, it appears, had found the tract in Latin in a *Cotton* MS. now in the British Museum. The next step was the discovery by Woepcke at Paris of an Arabic treatise on the division of figures. This he translated for the *Journal Asiatique*, 1851 (pp. 233 *et sq.*), and our author's next step is to compare this document with the *Practica Geometriae* of Leonardo Pisano (1220). Favaro's study of Leonardo's work is discussed, revised, and extended by Prof. Archibald, who gives his reasons for believing that Leonardo had access to some work of Greek-Arabic extraction. Here it is interesting to note how early the arithmetisation of geometry had begun.

The author fully acknowledges his indebtedness to Sir T. L. Heath's publications, as, indeed, must every future writer on Euclid. But there is ample evidence that in Prof. Archibald we have a writer of great erudition, equipped with the necessary critical faculty, and capable of doing work of the greatest value in a field in which English-writing workers are but few. This is, we believe, the first occasion on which a Greek mathematical text has been edited by one of our brethren across the pond. From its intrinsic value as a contri-

bution to our knowledge of the development of geometrical theory, and from its merits as a sound piece of historico-bibliographical work, we are amply justified in looking forward to work of still greater importance from hands of such proved competence. We had almost forgotten to draw attention to the valuable bibliography appended to this volume.

Analytic Geometry. By H. B. Phillips. Pp. vii + 197. 6s. 6d. net. 1915. (Wiley & Sons.)

Dr. Phillips writes for the student who is about to embark upon courses in the calculus and engineering, and proposes to supply him with the irreducible minimum of analytical geometry that is necessary for that purpose. Opening with a short recapitulation of some of the algebraical principles and processes that are presently required, the volume has about thirty pages given to rectangular coordinates, including a good little introduction to vectors. A useful novelty is the section on the sign of $Ax+By+C$, in which, by the way, "the value of the expression changes continuously *if the point P moves slowly*" is a curious statement. We remember no English text in which inequalities are dealt with as here, though the treatment, with diagram of shaded spaces, is common enough in continental text-books. The ellipse is defined as follows: "If a circle is deformed in such a way that the distances of the points from a fixed diameter are all changed in the same ratio, the resulting curve is called an ellipse. And a parabola is . . . a locus of points the squares of whose distances from one of two perpendicular lines are proportional to their distances from the other. The complete locus of such points is two parabolas." And in Art. 32 we have a similar definition for the hyperbola. It will be interesting to see if the lead here given by the author is followed by successors in the same field, as we may presume that he is justified by his experience as a teacher of the budding engineer in approaching the conics from this point of view. The treatment of "infinite values" is limited in scope and old-fashioned in form. There are useful sections on "empirical" equations, and parametric equations are not forgotten. The last thirty pages are devoted to the coordinates of a point in space, surfaces, lines as the intersections of planes, and curves as the intersections of surfaces. The exposition is clear throughout, but there seems to be some uncertainty in the equipment expected from the student, who at one moment is carried forward laboriously step by step, and at another has to take similar obstacles in his stride. But the book is well worth a place on shelves of reference for those who require such a text-book, or are anxious to get hints for their own work with students of the like class in this country.

Diophantine Analysis. By R. D. Carmichael. Pp. vi + 118. 5s. 6d. net. 1916. (Wiley & Sons; Chapman, Hall.)

De Morgan used to lament that the majority of those who attacked the problem of the quadrature of the circle did not seem to realise in what the nature of the problem consisted. It did not seem to occur to them that even if it were true that a large sum of money had been awarded for a solution, the supposed fact ought to show that there must be some subtlety inherent in the problem which had hitherto eluded the patience and sagacity of past generations. The same might be said on a smaller scale of the famous Last Theorem of Fermat, for the solution of which a valuable prize has been offered by Woolfskehl, the German mathematician. Those who read with sufficient care pp. 86-103 of the volume before us will have little excuse for supposing that a solution lies within the grasp of the untrained. The present monograph is designed to present the student with a systematic treatment of the Diophantine analysis. The successive chapters deal with: rational triangles, the method of infinite descent; problems involving a multiplicative domain; equations of the third, fourth, and higher degrees in two or more variables; and the method of functional equations. The author is familiar with the most important literature of the subject, but he does not seem aware that Desboves developed the method which he discusses on pp. 50 *et sq.* as partially used by Legendre. It would have added to the utility of the book had he added the sources from which his illustrative exercises are drawn; as he

generally gives the name and date, it would have been little more trouble to add the journals, etc., in which they were printed. For instance, we note the name of Gérardin on the last page. No doubt the theorem there attached to his name originally appeared in the *Intermédiaire* or some similar periodical, but a reference to *Sphinx-Oedipe*, the useful little journal edited by M. Gérardin, would introduce the student to a veritable storehouse of theorems connected with those other names with which he will be familiar by the time he has worked through the book, *e.g.* Aubry, Lucas, Maillet, Fauquembergue, Cunningham, Vérebrussov, etc. One more point. The private student will find himself at a considerable disadvantage in tackling the exercises, especially as, in this subject of all subjects, he requires the satisfaction of knowing that his solution is expressed in the most suitable form. Prof. Carmichael has accomplished his task with success. Considered as a whole, the book fills a gap. The presentation is clear, the arrangement admirable, and all that matters in a preliminary survey of the ground is to be found in either text or exercises. The book is well printed and "got up," and it seems to be quite free from misprints.

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Sir George Greenhill has presented a complete set of the *Bulletin of the American Mathematical Society*, and the thanks of the Association are due to him as well as to the many other donors whose gifts have been acknowledged from time to time in the *Gazette.*

ERRATA.

viii. No. 123, p. 257, l. 4 up,

$$\textit{for } \frac{23{\cdot}1\dot{6}}{10^2} \textit{ read } \frac{23{\cdot}1\dot{6}}{10^2}x.$$

viii. No. 124, p. 298, l. 10,

$$\textit{for } \log\Big(\textit{ write } \log\Big\{\Big($$

$$\textit{and for } \Big)\Big] \textit{ write } \Big)\Big\}\Big],$$

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